Kepler problem and Lorentz transformations

Based on [G. Meng, J. Math. Phys. **53**, 052901(2012)]

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God is a mathematician of a very high order — P. Dirac

What is Kepler Problem?

- It is a mathematical model for the simplest solar system.
- I. Newton introduced and solved it in 1678, and that leads to a good explanation for Kepler's three laws of planetary motion.
- It is also a mathematical model for the simplest atom (i.e. the hydrogen atom).
- E. Schrödinger introduced and solved it (at the quantum level) in 1926, and that leads to a good explanation for the spectral lines of the hydrogen gas and Mendeleev's periodic table for elements as well.
- It is a classical example of combining beauty, simplicity and truth all in one.

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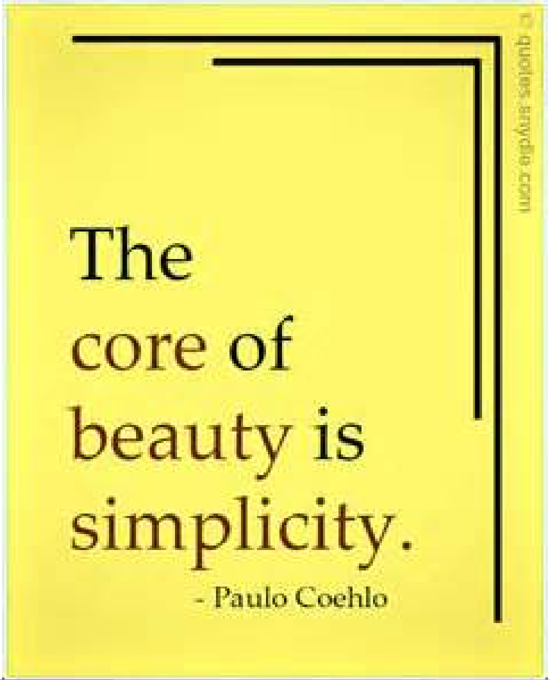
The
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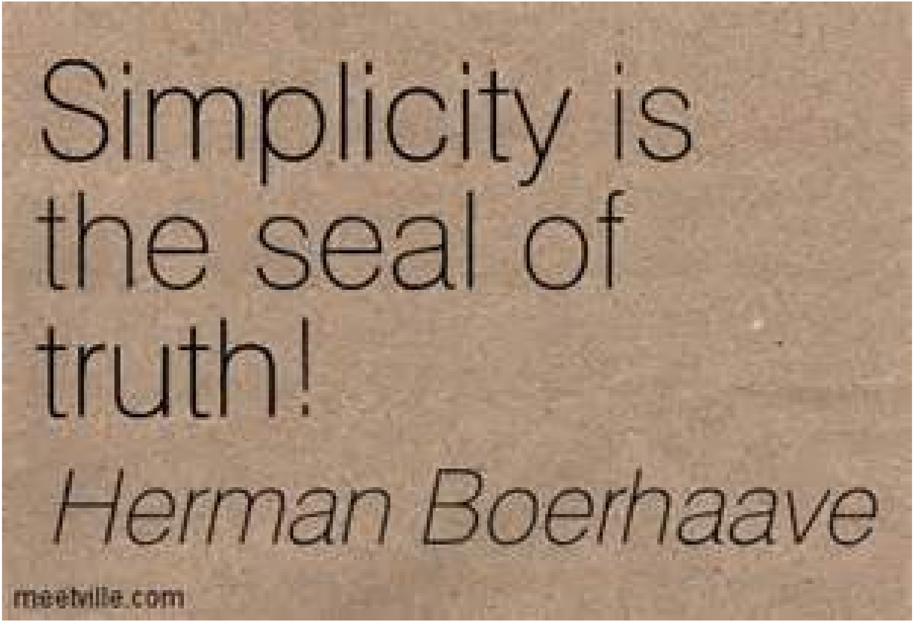
beauty is
simplicity.
- Paulo Coehlo


Simplicity is
the seal of
truth!
Herman Boerhaave
meetville.com

A few more quotes on beauty, simplicity and truth

1. *Beauty is the first test: there is no permanent place in the world for ugly mathematics*. — Godfrey Harold Hardy
2. *Everything should be made as simple as possible, but not simpler.* — Albert Einstein
3. *My work always tried to unite the truth with the beautiful, but when I had to choose one or the other, I usually chose the beautiful.* — Hermann Weyl

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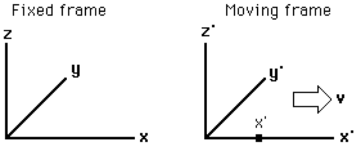
What are Lorentz Transformations?

- They are the linear transformations of the form

$$t = \gamma(t' + \frac{vx'}{c^2}),\ x = \gamma(x' + vt'),\ y = y',\ z = z'$$

for the time and (rectangular) space coordinates in two inertial frames:



Here c is the speed of light and $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$.

- They are the correction to the Galileo transformations:

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- They leave $c^2t^2 - x^2 - y^2 - z^2$ invariant.

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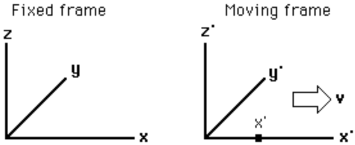

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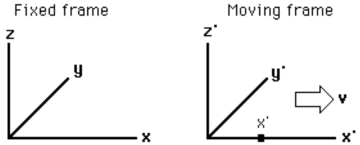

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- They (not Galileo transformations) leave invariant of the form of Maxwell equations for electromagnetism.
- The attempt to understand their geometric/physical meaning led to the relativity revolution in the early 20th century, including, among other things, the relativistic correction to Newtonian mechanics as well as various relativistic corrections to the Universal Gravitation Law, with Einstein's General Theory of Relativity being the favorite one.
- First published by Joseph Larmor in1897 and independently again by Hendrik Antoon Lorentz in 1899.
- In mathematics, any linear transformation T: $\mathbb{R}^4 \to \mathbb{R}^4$ that preserves the Lorentz inner product:

$$(x_0, \mathbf{x}) \cdot (y_0, \mathbf{y}) = x_0 y_0 - \mathbf{x} \cdot \mathbf{y}$$

is called a Lorentz transformation. Then Lorentz transformations form a group, i.e., the Lie group $\mathrm{O}(1, 3)$. For simplicity, we shall write x for $(x_0, \mathbf{x})$, x^2 for $x \cdot x$. So $x^2 = x_0^2 - \mathbf{x} \cdot \mathbf{x}$.

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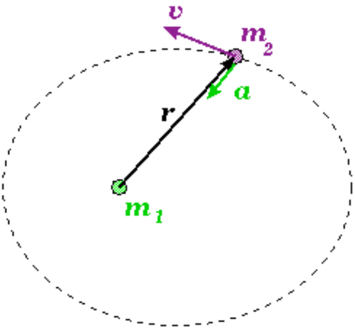
- Confuguration space: $\mathbb{R}^3_* := \mathbb{R}^3 \setminus \{\mathbf{0}\}$.
- Equation of Motion:

$$\mathbf{r}'' = -\frac{\mathbf{r}}{r^3}. \qquad (1)$$



- Angular Momentum $\mathbf{L} := \mathbf{r} \times \mathbf{r}'$ is conserved:

$$\mathbf{L}' = \mathbf{r}' \times \mathbf{r}' + \mathbf{r} \times \mathbf{r}'' = \mathbf{r} \times \left(-\frac{\mathbf{r}}{r^3}\right) = \mathbf{0}.$$

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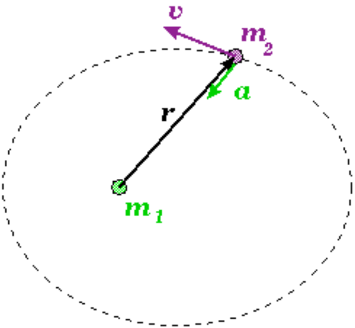

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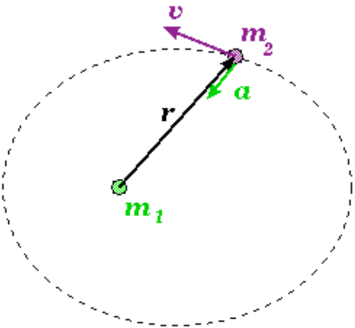

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- Lenz vector $\mathbf{A} := \mathbf{L} \times \mathbf{r}' + \frac{\mathbf{r}}{r}$ is conserved:

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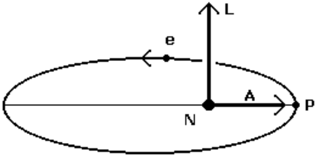
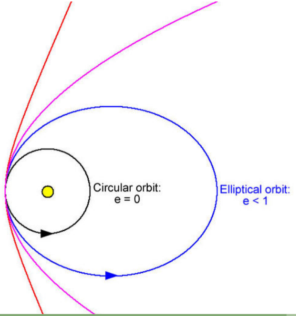
- Orbits. Since $\mathbf{L} = \mathbf{r} \times \mathbf{r}'$, $\mathbf{A} = \mathbf{L} \times \mathbf{r}' + \frac{\mathbf{r}}{r}$, we have $\mathbf{L} \cdot \mathbf{A} = 0$



and

$$\mathbf{L} \cdot \mathbf{r} = 0, \quad r - \mathbf{A} \cdot \mathbf{r} = |\mathbf{L}|^2, \tag{2}$$

So a non-colliding orbit is a conic with eccentricity e equal to $|\mathbf{A}|$:



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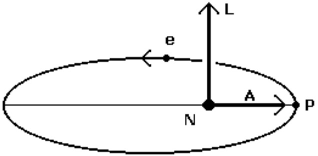

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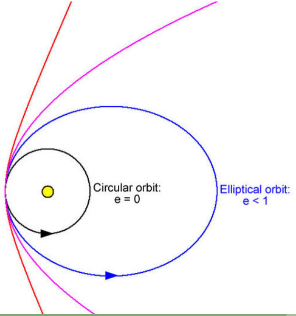

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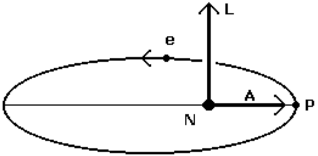

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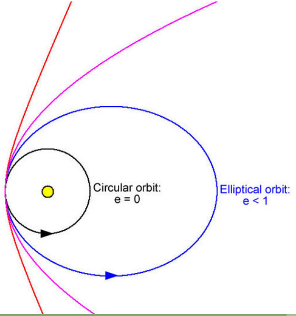

- Total energy. Assume the orbit is non-colliding (i.e. $\mathbf{L} \neq \mathbf{0}$), then the total energy $E := \frac{1}{2}|\mathbf{r}'|^2 - \frac{1}{r}$ can be expressed in terms of $\mathbf{L}$ and $\mathbf{A}$:

$$E = -\frac{1 - |\mathbf{A}|^2}{2|\mathbf{L}|^2}. \tag{3}$$

Proof.

$$\begin{aligned} |\mathbf{A}|^2 &= |\mathbf{L} \times \mathbf{r}'|^2 + 2\frac{\mathbf{r} \cdot (\mathbf{L} \times \mathbf{r}')}{r} + 1 \\ &= |\mathbf{L}|^2 |\mathbf{r}'|^2 - 2\frac{|\mathbf{L}|^2}{r} + 1 \\ &= 2|\mathbf{L}|^2 E + 1. \end{aligned}$$

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Magnetized Kepler Problems

- Magnetized Kepler problems were introduced towards the end of 1960s, by H. McIntosh and A. Cisneros, and independently by D. Zwanziger, so they are called MICZ-Kepler problems.
- They are the mathematical models for the hypothetical hydrogen atoms for which the nucleus carries magnetic charge as well.
- Their configuration spaces are all the same: $\mathbb{R}^3_* := \mathbb{R}^3 \setminus \{\mathbf{0}\}$.
- For the hypothetical hydrogen atom whose nucleus carries magnetic charge μ, its equation of motion is

$$\mathbf{r}'' = -\frac{\mathbf{r}}{r^3} - \mathbf{r}' \times \mu\frac{\mathbf{r}}{r^3} + \frac{\mu^2}{r^4}\mathbf{r} \tag{4}$$

  Conserved quantities are angular momentum $\mathbf{L} := \mathbf{r} \times \mathbf{r}' + \mu\frac{\mathbf{r}}{r}$ and Lenz vector $\mathbf{A} := \mathbf{L} \times \mathbf{r}' + \frac{\mathbf{r}}{r}$.
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- It is easy to see that $\mathbf{L} \cdot \mathbf{A} = \mu$, and

$$\mathbf{L} \cdot \mathbf{r} = \mu r, \qquad r - \mathbf{A} \cdot \mathbf{r} = |\mathbf{L}|^{\mathbf{2}} - \mu^{\mathbf{2}}. \tag{5}$$

- Eq. (5) gives an algebraic description for the orbits, from which, we deduce that there are four types of orbits: linear, elliptic, parabolic, and hyperbolic.

Exercise1： For conic(5), please express its eccentricity in terms of **L**, **A** and μ.

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A new description for the orbits

The preceding set of algebraic equations can be rewritten as

$$\mu r - \mathbf{L}\cdot\mathbf{r} = 0, \quad r - \mathbf{A}\cdot\mathbf{r} = |\mathbf{L}|^{\mathbf{2}} - \mu^{\mathbf{2}}. \tag{6}$$

Assume that the orbit is non-collding, i.e. $|\mathbf{L}|^2 - \mu^2 = |\mathbf{r}\times\mathbf{r}'|^2 > 0$. Then, we can introduce 4-D Lorentz vectors

$$l = \frac{1}{\sqrt{|\mathbf{L}|^{\mathbf{2}} - \mu^{\mathbf{2}}}}(\mu, \mathbf{L}), \quad a = \frac{1}{|\mathbf{L}|^{\mathbf{2}} - \mu^{\mathbf{2}}}(1, \mathbf{A}), \quad x = (r, \mathbf{r}) \tag{7}$$

so that Eq. (6) can be rewritten as

$$l\cdot x = 0, \quad a\cdot x = 1. \tag{8}$$

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Remark: Eq. (8) is for $\mathbf{r}\in\mathbb{R}^3_*$, but it is also for $x\in\mathbb{R}^4$ provided that x is on the future light cone.

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Assume that the orbit is non-collding, i.e. $|\mathbf{L}|^2 - \mu^2 = |\mathbf{r}\times\mathbf{r}'|^2 > 0$. Then, we can introduce 4-D Lorentz vectors

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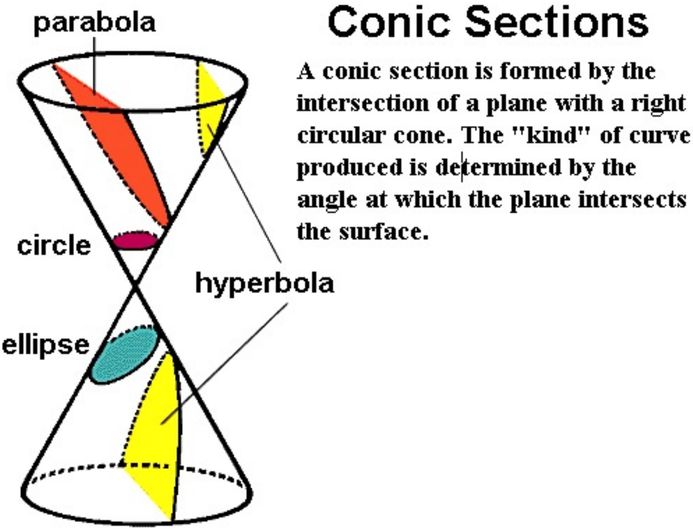
Exercise 2: Please describe the linear orbits on the future light cone.

Future light cone in the 3-D Lorentz space



Kepler Problem and Lorentz Transformations

MICZ-Kepler orbit — a non-colliding orbit in a MICZ-Kepler problem. There are three types: elliptic, parabolic, and hyperbolic.

little Lorentz transformation — a little linear transformation T from $\mathbb{R}^4$ to $\mathbb{R}^4$ which preserves the Lorentz inner product. Here "little" means that T can be continuously deformed to the identity map on $\mathbb{R}^4$.

scaling transformation — the scalar multiplication of vectors in $\mathbb{R}^4$ by a positive real number.

Theorem (G. Meng, J. Math. Phys. 53, 052901(2012))

1) Any two oriented parabolic MICZ-Kepler orbits can be transformed from one to the other via a little Lorentz transformation.
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Remark. 1) *A second temporal dimension (i.e. x_0) appears naturally*.
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